MATH-329 Continuous optimization
Exercise session 12: Constrained algorithms

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1. Implementation of quadratic penalty method. Write code to apply the quadratic penalty method to the simple optimization problem

$$\min_{x\in\mathbb{R}^2} x_1 + x_2 \quad \text{subject to} \quad x_1^2 + x_2^2 - 2 = 0.$$

To solve the subproblems, that is, to minimize F_β for each individual value β, you can use code you wrote in previous exercise sessions / homework assignments, or you can use Matlab's Optimization Toolbox: the code below requires you to provide a function `[val, grad] ... = F(x, beta)` implementing $F_\beta(x)$ (as the first output) and $\nabla F_\beta(x)$ (as the second output) as well as an initialization `x_in`, and it attempts to return a minimizer `x_out`.

```
% See 'help fminunc': Matlab's unconstrained minimizer.
options = optimoptions('fminunc', 'SpecifyObjectiveGradient', true);
x_out = fminunc(@(x) F(x, beta), x_in, options);
```

It is instructive to visualize the penalized function F_β for various values of β to get a sense of how the penalty shapes the 'landscape' of the cost function, and to display on those plots the sequence of solutions (x_k) that you compute.

2. Unbounded penalized function. The function F_β from the previous exercise may fail to be bounded below, in which case minimizing it can completely fail. Verify this claim on the following example:

$$\min_{x\in\mathbb{R}^2} -5x_1^2 + x_2^2 \quad \text{subject to} \quad x_1 = 1.$$

Argue that for $\beta < 10$ the function F_β is unbounded below. What happens for $\beta \geq 10$? Can we still hope to find a solution for this problem via some instantiation of the quadratic penalty method?